On Validated Numeric of Values of Some Zeta and L-functions, and Applications

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Dedicated to Hans J. Stetter on his 70th birthday

1 Introduction

This note is for an introduction to the validated numerics study of values of zeta and L - functions. The subject matter of this talk lies in the area between Analysis, Algebra and Validated Numerics [1, 7].More specifically, I want to discuss the relations between the algebra and analysis of zeta and L - functions and the evaluation of values of the functions. The talk is on some observations and partial results related to more general cases, and I would like to ask the permission of the listeners for replacing the notes of my talk by a very introductory exposition. The reason for this "going down" and taking very elementary examples as the object for discussion lies on my side, i.e. it was useful for the study of more general cases to review the simplest cases.
Why Zeta and L-functions are considered?
As noted by Don Zagier, zeta and L-functions of various sorts are all-pervasive objects in modern number theory and its applications [8]. They appear in number theory, in algebraic geometry, in algebraic K-theory, in representation theory and in the theory of automorphic forms. Recently zeta and L-functions appear in the theory of dynamical systems [18, 19]. Many mentioned applications connected with computations of zeta and L-functions values in "critical" points (following Deligne [10] an integer m is called critical if neither m nor $h-m$ is a pole of $\gamma(s)$, where $\gamma(s)$ is a "gamma factor", which equal to an exponential function A^s times a finite product of terms $\Gamma(1/2(s+m))$ such that the product $L^*(s) = \gamma(s)L(s)$ satisfies $L^*(s) = wL^*(k-s)$ for some integer $h > 0$ and sign $w = \pm 1$). Then the corresponding critical value $L(m)$ should have the form

$$L(m) = A(m)\Omega(m) \ (m \text{ critical})$$

where $\Omega(m)$ is a predictable period (the integral over some algebraic cycle of an algebraic differential form defined over a number field) and $A(m)$ is an algebraic number belonging to a predictable number field.

The organization of this note is as follows: In Section 1 we remind definitions and some properties of Riemann zeta function, Dirichlet L-functions, Dedekind zeta function, Artin-Hasse zeta functions and Hasse-Weil zeta function of elliptic curves. Section 2 comments some applications of zeta and $L-$functions. In Section 3 we consider some validated numerics aspects of zeta and $L-$functions and interval evaluation of Riemann zeta function for $Re(s) \geq 1$ and values of some Dirichlet L-functions at point $s = 1$. Also we consider some aspects of interval evaluation of Hasse-Weil zeta functions of elliptic curves at point $s = 1$.

2 Zeta and L - functions

Let us remind definitions and some properties of Riemann zeta function, Dirichlet L-functions, Dedekind zeta function, Artin-Hasse zeta functions and Hasse-Weil zeta function of elliptic curves.

2.1 Riemann's zeta function

Let

$$\sum_n \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1}, n \in \mathbf{N},\ p = 2, 3, 5, 7, \ldots (primes) \tag{1}$$

be the Euler product. Euler used this formula principally as a formal identity and principally for integer values of s. Let

$$\zeta(s) = \sum_n \frac{1}{n^s}. \tag{2}$$

Dirichlet, unlike Euler, used the formulae (1.1), (1.2) with s as a real variable, and, also unlike Euler, he proved rigorously that (1.1) is true for all real $s > 1$. Riemann defines the function [24, 12]

$$\zeta(s) = \frac{\Gamma(-(s+1))}{2\pi i} \int\limits_{+\infty}^{+\infty} \frac{(-x)^s}{exp(x) - 1} \frac{dx}{x}, \tag{3}$$

where $\int\limits_{+\infty}^{+\infty} \ldots$ is a counter integral. The limits of integration are intended to indicate a path of integration which begins at $+\infty$, moves to the left down the positive real axis, circles the origin once in the positive (counterclockwise) direction, and returns up the positive real axis to $+\infty$. If $\zeta(s)$ is defined by formula (1.3), then for real values of s greater than one, $\zeta(s)$ is equal to function (1.2). The formula (1.3) defines a function $\zeta(s)$ which is analytic at all points of the complex $s-$plane except for a simple pole at $s = 1$. The function $\zeta(s)$ satisfies the functional equation

$$\zeta(s) = \Gamma(-(s+1))(2\pi)^{s-1} 2\sin(s\pi/2)\zeta(1-s). \tag{4}$$

Let B_k be the k-th Bernoulli number in even numeration $B_0 = 1, B_1 = -\frac{1}{2}, \ldots$. For $n \in \mathbf{Z}$ Euler have found that

$$\zeta(2n) = \frac{(2\pi)^{2n}(-1)^{n+1}B_{2n}}{2(2n)!}.$$

If $n > 0$ then $\zeta(-2n) = 0$. Functional equation (1.4) gives for $m \in \mathbf{N}$ that

$$\zeta(1-2m) = (-1)^{1-2m}\frac{B_{2m}}{2m}.$$

2.2 Dirichlet L-functions

Let p be a prime. Let $\omega^{p-1} = 1$. Let g be a primitive root $\pmod p$. Let $\nu(n)$ be the index of natural n relatively to the primitive root g, i.e. $g^{\nu(n)} \equiv n \pmod p$.
Elementary Dirichlet character to a $\pmod p$ is the function

$$\chi(n) = \omega^{\nu(n)},\ \chi(n) = 0 \text{ if } p \mid n.$$

There are $\varphi(p) = p-1$ distinct functions: $\chi(n) = e^{\frac{2\pi i m}{p-1}}$, $1 \le m \le p-1$. Let s be a positive real. Let

$$L_\omega(s) = \sum_{n=1}^{\infty} \chi(n) n^{-s}. \tag{5}$$

The function $L_\omega(s)$ is called a Dirichlet L-function.
Let $\chi(n)$ be a Dirichlet character to a $\pmod q$. In the case there exists a Dirichlet L-function $L(s,\chi)$ [9]. Let $e_q(m) = e^{\frac{2\pi i m}{q}}$. Let

$$\tau(\chi) = \sum_{m=1}^{q} \chi(m) e_q(m)$$

be a Gauss sum. Let now χ be a primitive character, and let [9]

$$\alpha = \begin{cases} 0, & \text{if } \chi(-1) = 1 \\ 1, & \text{if } \chi(-1) = -1 \end{cases}$$

Let

$$\xi(s,\chi) = (\frac{\pi}{q})^{(-1/2)(s+\alpha)}\Gamma(\frac{1}{2}(s+\alpha))L(s,\chi).$$

Then

$$\xi(1-s,\overline{\chi}) = \frac{i^\alpha q^{1/2}}{\tau(\chi)}\xi(s,\chi) \tag{6}$$

is the functional equation for $L(s,\chi)$. Dirichlet $L-$functions are a partial case of Dirichlet series.

2.3 Dedekind zeta functions

Dedekind extended the Riemann zeta function from natural numbers n to all integer divisors (ideals) of an algebraic number field K. The Dedekind zeta function $\zeta_K(s)$ is defined by the formula

$$\zeta_K(s) = \sum_a \frac{1}{N(a)^s}. \tag{7}$$

Here a runs over all integer divisors (ideals) of the field K and $N(a)$ is the norm of a.
Let $m \in \mathbf{N},\ \xi^m = 1$. A field K is the $m-$cyclotomic field if $K = \mathbf{Q}(\xi),\ \xi^m = 1$. Let $m = p$, where p is a prime. In the case

$$\zeta_K(s) = G(s)\prod_\chi (1 - \frac{\chi(p)}{p^s})^{-1}.$$

Here $G(s) = \prod_{\rho|p}(1 - \frac{1}{N(\rho)^s})^{-1}$ and ρ runs over all prime divisors of the field $\mathbf{Q}(\sqrt[p]{1}),\ \rho \mid p$. Let χ be all Dirichlet characters $\pmod p$.
Proposition. [4] Suppose K is the $p-$cyclotomic field. Then the Dedekind's zeta function of K is the product of $G(s)$ and $\varphi(p)$ Dirichlet $L-$functions runs over all Dirichlet characters χ :

$$\zeta_K(s) = G(s)\prod_\chi L(s,\chi).$$

Let p be a positive integer, $p \equiv 1 \pmod 4$, and $K = \mathbf{Q}(\sqrt{p})$. From Siegel's [21] follows, that

$$2\zeta_K(-1) = \frac{1}{15}\sum_{1\le b<\sqrt{p},\, b\, odd} \sigma_1(\frac{p-b^2}{4}),$$

where $\sigma_1(n)$ is the sum of the divisors of n.

2.4 Zeta and L-functions of elliptic curves

[23] Let $E/\mathbf{Q}$ be an elliptic curve given in Weierstrass form by an equation

$$E: y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \tag{8}$$

and let $b_2, b_4, b_6, b_8, c_4, c_6, \Delta, j$ be the usual associated quantities [23]. Let now (8)be a global minimal Weierstrass equation for E over $\mathbf{Z}.$ For each prime p the reduction (8) $\pmod p$ defines a curve E_p over the prime field $\mathbf{F}_p$. Let A_p denote the number of points of E_p rational over $\mathbf{F}_p$. Let

$$t_p = 1 + p - A_p.$$

If $p \not| \Delta$, then t_p is the trace of Frobenius and satisfies $|T_p| \leq 2\sqrt{p}$. In the case *Artin-Hasse zeta function* of the elliptic curve E_p is:

$$\zeta_{E_p}(s) = \frac{1 - t_p p^{-s} + p^{1-2s}}{(1-p^{-s})(1-p^{1-s})}. \tag{9}$$

If $p \mid \Delta$, then E_p is not an elliptic curve and has a singularity S. In the case

$$t_p = \begin{cases} 0, & \text{if } S \text{ is a cusp,} \\ 1, & \text{if } S \text{ is a node,} \\ -1, & \text{if } S \text{ is a node with tangent quadratic over } \mathbf{F}_p \end{cases}$$

The *Hasse-Weil* $L-$function of $E/\mathbf{Q}$ is defined by equation

$$\text{Ł}_E(s) = \prod_{p \mid \Delta} \frac{1}{(1 - t_p p^{-s})} \prod_{p \not| \Delta} \frac{1}{1 - t_p p^{-s} + p^{1-2s}} \tag{10}$$

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3 Evaluation of values of zeta and L - functions and their applications

Many results in number theory are based on evaluation of values of Riemann zeta and Dirichlet L-functions (zeros of zeta and L-functions, prime number theorem, distribution of primes in arithmetic progressions, density theorems). Here we remind some another applications (please, see also very interesting paper [8]).

Evaluation of the index of elliptic operator and zeta functions [20, 2]

Let A be a self adjoint positive pseudo-differential (Calderon-Zygmund) operator. Let A has no eigenvalues λ on negative real axis. We can define

$$A^s = \frac{1}{2\pi} \int_\Gamma \lambda^s (A - \lambda)^{-1} d\lambda,$$

where Γ is the counter which begin at $-\infty$, moves to the right up the negative real axis, circles the origin once in the clockwise direction, and returns down the negative real axis to $-\infty$.
Let $A = -\frac{d^2}{dx^2} + P$ be the operator on unit circle S^1, $Pf = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx$. Eigenvalues of A are natural numbers $1, 1, 1, 4, 4, \dots, m^2, m^2, \dots$ and if λ_j is the j eigenvalue of A, then $trA^s = Z(s) = 1 + 2\sum_1^\infty \lambda_j^s = 1 + 2\zeta(-2s)$. Suppose now that D is an elliptic operator and D^* is the adjoint of D. Then

$$\Delta_0 = 1 + D^*D, \ \Delta_1 = 1 + DD^*$$

are positive self-adjoint operators. Let

$$\zeta_i(s) = tr\Delta_i^{-s}, i = 0, 1.$$

Then the difference of ζ_0 and ζ_1 gives the index of the operator D.

The symplectic volume of moduli space [11, 26]

Let $\mathcal{M}(n, d)$ be the moduli space of (semi)stable holomorphic vector bundles of rank n, degree d and fixed determinant on a compact Riemann surface Σ. The symplectic volume of $\mathcal{M}(2, 1)$ is given by

$$vol(\mathcal{M}(2, 1)) = (1 - \frac{1}{2^{2g-3}})\frac{\zeta(2g-2)}{2^{g-2}\pi^{2g-2}}.$$

Here g is the genus of Σ. If n, d, g are sufficiently large, then an evaluation of $vol(\mathcal{M}(n, d))$ is desirable.

Generalized Riemann conjecture

Let $s = \sigma + it$. Let $L(s, \chi)$ be a Dirichlet $L-$function. Then

$$L(s, \chi) \neq 0 \text{ if } \sigma > \frac{1}{2}. \qquad (11)$$

The Hilbert modular group and the volume of orbifolds [16]

Let K be a real quadratic field over $\mathbf{Q}$ and R the ring of algebraic integers in K. Let H be the upper half plane of $\mathbf{C}$. The group $SL(2, R)$ acts on $H \times H$. The Hilbert modular group $\Gamma = SL(2, R)/(1, -1)$ acts on $H \times H$ effectively. The volume of $X = \Gamma\backslash(H \times H)$ is given by

$$vol(X) = 2\zeta_K(-1).$$

The Birch and Swinnerton-Dyer conjecture for elliptic curves [23, 6, 5]

The Hasse-Weil $L-$function (10) converges for $Re(s) > \frac{3}{2}$ because it is dominated by the product $(\zeta(s - \frac{1}{2}))^2$.
The Birch and Swinnerton-Dyer conjecture for elliptic curves is
BSDC1 Let r be the rang of $E(\mathbf{Q})$. Then $L(E, s)$ has a zero of order r at $s = 1$. In particular, $L(E, s)$ has the Taylor expansion

$$L(E, s) = C_E(s-1)^r + \text{ higher terms with some constant } C_E \neq 0.$$

The second Birch and Swinnerton-Dyer conjecture for elliptic curves concerns this constant. Arithmetic investigations involve derivatives of $L(E, s)$ and computation of their values [15].

4 Validated numerics of values of zeta and L - functions

4.1 Validated numerics and zeta functions

Remind at first the interval interpretation of inequality $f(z) \neq 0$. It appears in formula (11) and in many another cases in the theory of zeta and $L-$functions. The inequality has the interpretation:
(i) If $f(z) \in \mathbf{R}$ then $(f(z) > 0) \vee (f(z) < 0)$ and $f(x) \in A$ for a real interval $A, 0 \notin A$.
(ii) If $f(z) \in \mathbf{C},\ f(z) \notin \mathbf{R}$ and we use complex rectangular arithmetic $I\mathbf{C}$ [1], then $(Re(z) > 0) \vee (Re(z) < 0) \vee (Im(z) > 0) \vee (Im(z) < 0)$. So $f(z) \in A,\ A \in I\mathbf{C},\ 0 \notin A$. Side by side with rectangular arithmetic there are complex circular arithmetic [1] and complex sector arithmetic [17]. In some cases they are very useful for evaluation of $f(z) \neq 0$. But as this requires the consideration of numerous different cases we omit it. In preceding sections we have reminded some results about values of zeta functions at integer (critical) points. A common problem of validated numerics to find optimal interval evaluation of the value of a function with interval variables and parameters. In many cases for the purpose we can use:

(i) programming languages and compilers for scientific computations (Pascal-XSC, ACRITH-XSC, Fortran-XSC, Oberon-XSC);
(ii) computer algebra systems with interval packages;
(iii) highly accurate, extended computer arithmetics with standard and special functions for real and interval data [7].

H. Stetter gave in [22] and in his another papers applications of computer algebra in interval methods. In the frame of zeta and $L-$functions we can apply in some cases algebraic-analytic approach and corresponding computer algebra for exact presentation of zeta or $L-$function as a finite sum and a remaining term. Then the sum and the remaining term are interval evaluated [14]. Computer algebra systems include now in their standard library functions some kinds of Zeta functions. For instance Maple V Release 4 includes JacobiTheta1-Theta3 (Theta functions are connected with Zeta functions), WeierstrassZeta, Riemann Zeta and Hurvitz Zeta. So we can in most interesting cases compute (not guaranteed) values of the functions. In some cases the Maple computing $Zeta(n, a, m), n \in \mathbf{Z}, a, m \in \mathbf{Q}$ in closed form (by $\pi, a, \ln$). Standard functions and packages of Maple give possibility to compute values of some Zeta and L-functions that do not include in standard Maple library. In any case we have to interval evaluate the expression for zeta or $L-$function or the closed form of value of zeta or $L-$function.

4.2 Interval evaluation

4.2.1 Riemann zeta

Consider the series $\zeta(s) = \sum_n \frac{1}{n^s}$ for complex values of s with $Re(s) \geq 1$. Interval evaluation of the $\zeta(s)$ can be taken from the result of Backlund (a little bit reformulated). The result is based of Euler-Maclaurin summation.
Proposition (Backlund) Let N be natural > 1. Let $s = \sigma + it$ and let $\sigma \geq 1$. Let B_{2k} be the Bernoulli numbers in even numeration,
$S(N-1, s) = \sum_{n=1}^{N-1} n^{-s} + \frac{N^{1-s}}{s-1}$,
$B(N, k, s) = \frac{1}{2}N^{-s} + \frac{B_2}{2} s N^{-s-1} + \cdots + \frac{B_{2k}}{(2k)!} s(s+1)\ldots(s+2k-2)N^{-N-2k+1}$.
Then

$$\zeta(s) = S(N-1, s) + B(N, k, s) + R_{2k}, \tag{12}$$

where

$$|R_{2k-2}| \leq \left| \frac{s+2k-1}{\sigma+2k-1} \right| |B_{2k} \text{ term of (12)}|.$$

4.2.2 Dirichlet series

Let $K = \mathbf{Q}(\sqrt{D})$ be a real quadratic field with positive integer squarefree D and $\chi(n) = \left(\frac{\Delta}{n}\right)$ be the Kronecker symbol. Here

$$\Delta = \begin{cases} D, & D \equiv 1 \pmod 4, \\ 4D, & D \equiv 2, 3 \pmod 4). \end{cases}$$

Let $A = \pi/\Delta,\ E(x) = \int_x^\infty \frac{e^{-t}}{t} dt,\ erfc(x) = \frac{2}{\pi} \int_x^\infty e^{-t^2} dt.$
Proposition [25, 3].

$$L(1, \chi) = \frac{1}{\sqrt{\Delta}} \sum_{n=1}^{m} \chi(n) E(An^2) \ + \sum_{n=1}^{m} \left(\frac{\chi(n)}{n} \right) erfc(n\sqrt{A}) + R_m,$$

where $|R_m| < \frac{\Delta^{3/2}}{\pi^2} \frac{e^{-Am^2}}{m^3}$.

4.2.3 Remarks about $L-$functions of elliptic curves.

From the work of A. Wiles, R. Taylor and A. Wiles, and work of F. Diamond it is known that (semistable) elliptic curves over $E/\mathbf{Q}$ are modular. Knowing the modularity of $E/\mathbf{Q}$ is equivalent to the existence of a modular form f on $\Gamma_0(N)$ for some natural value N, which we write $f = \sum a_n q^n$. The $L-$function of E is thus given by the Mellin transform of f, $L(f, s) = \sum a_n q^n$. In particular, the behavior of $L(E, s)$ at $s = 1$ can be deduced from modular properties of E.
Let $E/\mathbf{Q}$ be a modular elliptic curve and the global minimal model of the $E/\mathbf{Q}$ has prime conductor l. Let p be a prime and A_p (as in (1.4)) be the number of points of E_p in $\mathbf{F}_p$. Then there is exists a modular form f on $\Gamma_0(l)$, $f = \sum_{n=1}^{\infty} a_n q^n$, where a_p, $p \neq l$, equals $p + 1 - A_p$. Under these assumptions it seems that results of [5] gives expressions for $L(E, 1)$ and $L^{'}(E, s)$, which can evaluate by validated numerics methods.

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